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LETTER TO THE EDITOR

Magnetization of a Potts ferromagnet on a Sierpinski carpet

Anna Chame† and Uriel M S Costa‡

† Instituto de Física, Universidade Federal Fluminense, Outeiro S João Batista s/n°, Niterói, RJ, Brazil

‡ Departamento de Física, Universidade Federal de Alagoas, 57000 Maceió, AL, Brazil

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Abstract. We study the inhomogeneous magnetization behaviour of a q-state Potts ferromagnet on a Sierpinski carpet as a function of the temperature, using a real-space renormalization group procedure. Two qualitatively different regions on the carpet are distinguished: (a) the interior of the carpet and (b) the set of internal walls which border the eliminated areas. The spontaneous mean magnetization curves for these regions are obtained with the corresponding critical exponents β and β_W, for different values of q. We find that the magnetization of region (b) goes to zero at the critical temperature always below the one of region (a).

To establish how the fractal nature of a body alters its physical properties is an active field of interest nowadays. In particular the study of critical phenomena on fractal lattices (as they could simulate the percolative cluster on a phase transition) has been object of much attention (Gefen *et al* 1980, 1981, 1983a, 1984, Havlin *et al* 1983, Suzuki 1983, Bhanot *et al* 1984, 1985). It has been shown (Gefen *et al* 1980) that, for Ising systems, the presence of an ordered phase on these fractal lattices is associated with an infinite order of ramification (Mandelbrot 1982), which is the case of the Sierpinski carpet family.

Gefen *et al* (1984) treated Ising models on these carpets using a bond-moving renormalization group (RG) scheme, by considering that the iteration of the RG transformation generates two basic exchange variables, which implies the establishment of two recursion relations. A similar point of view, using RG techniques, was adopted by Costa *et al* (1987) to make a quantitative analysis of the q-state Potts ferromagnet, this time simulating the Sierpinski carpet by appropriate hierarchical lattices (Berker and Ostlund 1979, Griffiths and Kaufman 1982, Melrose 1983a, b, Kaufman and Griffiths 1984, Tsallis 1985). This scheme gave good numerical estimates for the criticality of Sierpinski carpets, showing that the transformation used is quite satisfactory.

The purpose of this letter is to study how the magnetization on a Sierpinski carpet behaves, by using the same transformations as Costa *et al* (1987). As the magnetization on a fractal lattice (which is not translationally invariant) is non-uniform, we consider, as an approximation, two typical regions on the carpet and calculate the corresponding mean magnetizations (as functions of the temperature) and critical exponents β, using a RG procedure introduced by Caride and Tsallis (1987). In what follows, we first discuss the RG transformation. The results for the magnetization curves and critical exponents β are then presented and discussed. Finally we present our conclusions.

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Consider a Sierpinski carpet obtained by dividing a unit square into b^2 small squares and then cutting out symmetrically l^2 central squares (as shown in figure 1, with $b=3$ and $l=1$). This operation is carried out iteratively on the microscopic scale. The fractal dimension of the lattice obtained in this way is $D_f=(b^2-l^2)/\ln(b)$. We attach to each site on this fractal lattice a q-state Potts variable. The Hamiltonian is given by:

$$H=-\sum_{\langle i,j\rangle} J_{ij}\delta_{\sigma_i,\sigma_j} \tag{1}$$

where the sites interact ferromagnetically through $J_{ij}=J_W$ if both sites i and j are on the walls of eliminated areas on the fractal and through $J_{ij}=J$ otherwise.

The RG procedure of Caride and Tsallis (1987) allows us to calculate the order parameter for this system directly (without going through the calculation of the thermodynamic free energy). We summarize it here, as we apply it to our model.

Initially we consider a Sierpinski carpet of linear size L. We associate each site on this microscopic fractal lattice with an elementary magnetic moment, μ, which we leave as a new variable of the RG procedure (the other variables are $K=J/k_BT$ and $K_W=J_W/k_BT$). We define the order parameters in the thermodynamic limit $L\to\infty$ for (a) the system consisting of all sites which are *not* on the walls bordering the eliminated areas, as $M=N(K,K_W)/L^D$ and for (b) the system consisting of all sites on the walls, as $M_W=N_W(K,K_W)/L^{D_W}$. $N(K,K_W)$ is the thermal average number of sites in system (a) with the Potts variable in a privileged state minus those with Potts variable in any other state (analogously, $N_W(K,K_W)$ corresponds to system (b)). D and D_W are such that $L^D\sim M$, the mass (number of sites) of system (a) and $L^{D_W}\sim M_W$, the mass (number of sites) of the system (b). We mention that, for the Sierpinski carpet we considered, D and D_W are the same ($D=D_W=D_f$).

The original system is transformed into a renormalized carpet of linear size L', with renormalized variables K', K'_W, and μ'. We propose that, through renormalization, both the total magnetic moments of systems (a) and (b) must be preserved (as they are extensive quantities):

$$N(K',K'_W)\mu'=N(K,K_W)\mu \tag{2a}$$

$$N_W(K',K'_W)\mu'=N_W(K,K_W)\mu. \tag{2b}$$

Following along the lines of Caride and Tsallis (1987) we obtain for (K,K_W) belonging to the ordered phase:

$$M(K,K_W)=\lim_{n\to\infty}\frac{\mu^{(n)}}{B^{nD}} \tag{3a}$$

$$M_W(K,K_W)=\lim_{n\to\infty}\frac{\mu^{(n)}}{B^{nD_W}} \tag{3b}$$

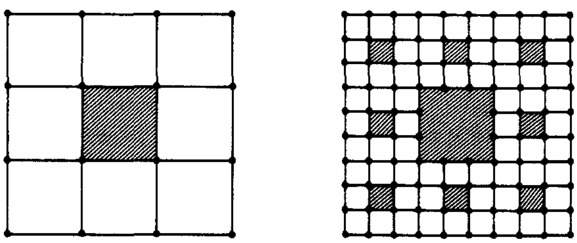

Figure 1. Iteration of the Sierpinski carpet. (a) Initial unit square with $b=3$, $l=1$. (b) The second iteration.

where $B = L/L' > 1$ is the linear scale factor and n is the number of iterations of the RG transformation. If the point (K, K_W) belongs to the disordered phase, we obtain that both $M(K, K_W)$ and $M_W(K, K_W)$ are equal to zero, as expected.

To close the procedure we now specify how to generate the RG recurrence relations for K, K_W, and μ. We shall adopt the same aggregation procedures as Costa *et al* (1988), for the Sierpinski carpet with $b = 3$ and $l = 1$, shown in figure 2. The neighbourhood of the interface between two cells (figure 2(*a*)) is associated with the renormalization of the K bond and the neighbourhood of the interface between a cell and an eliminated cell (figure 2(*d*)) is associated with the renormalization of the K_W bond. In figures 2(*b*) and 2(*e*) we show how to obtain the hierarchical lattices which we will use to simulate each one of these regions. In our case these hierarchical lattices are of the kind called by Griffiths and Kaufman (1982) 'non-uniform': they mix different kinds of bonds, each with its own scheme of aggregation.

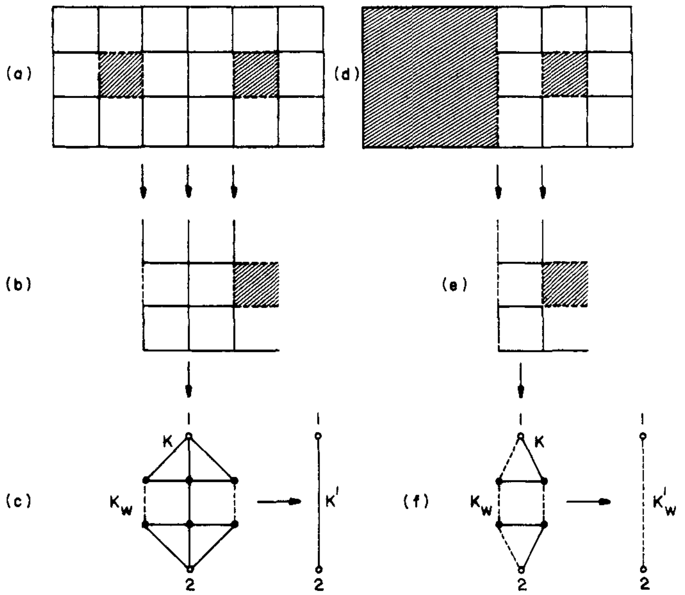

Figure 2. Construction of a hierarchical lattice adequate to simulate the Sierpinski carpet with $b = 3$, $l = 1$. (*a*), (*b*) and (*c*) show how K' is obtained. (*d*), (*e*) and (*f*) show how to obtain K'_W. Full lines denote the coupling constant K whereas broken lines denote K_W.

As we are now simulating the two regions on the carpet by hierarchical lattices, the factors B^D and B^{D_W} which appear in equations 3(*a*) and 3(*b*) will be approached (see Chame *et al* 1987) by B^d and B^{d_W}, respectively, where:

$$B^d = \frac{R + R_W K_W/K}{R' + R'_W K'_W/K'} \tag{4a}$$

$$B^{d_W} = \frac{S + S_W K_W/K}{S' + S'_W K'_W/K'}. \tag{4b}$$

R and R_W are respectively the number of K and K_W bonds in the largest graph of figure 2(*c*) while S and S_W are respectively the number of K and K_W bonds in the largest graph of figure 2(*f*). The primes indicate the renormalized graphs, in an analogous way.

The renormalization equations for K and K_W have been obtained (Costa *et al* 1987) through the break-collapse method (Tsallis and Levy 1981), using the more suitable variables t and t_W:

$$t = (1-\exp(-qJ/k_BT))/(1+(q-1)\exp(-qJ/k_BT)) \quad (5a)$$

$$t_W = (1-\exp(-qJ_W/k_BT))/(1+(q-1)\exp(-qJ_W/k_BT)). \quad (5b)$$

These equations have the form:

$$t' = f(t, t_W) \quad (6a)$$

$$t'_W = g(t, t_W) \quad (6b)$$

and give a phase diagram (Costa *et al* 1987) as shown in figure 3. In this figure, A and C are, respectively, the trivial stable fixed points corresponding to the disordered and ordered phases.

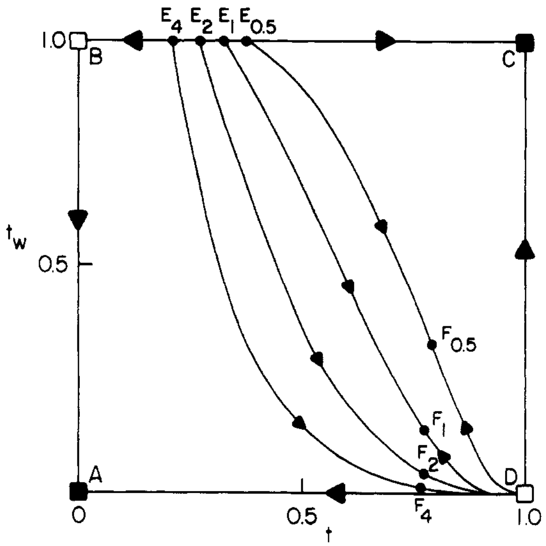

Figure 3. Phase diagram (Costa *et al* 1987) in $t-t_W$ space for $q = 0.5$, 1, 2 and 4, obtained through the RG transformation indicated in figure 2. The points A, B, C, D, E, F are fixed points whose stabilities are given by the arrows.

We need extra equations, besides the ones for K' and K'_W, to calculate the 'effective' magnetic moments for the two renormalized clusters, which correspond to the two typical regions on the carpet. These equations are obtained by requiring that the total magnetic moment of the original and renormalized clusters for both transformations in figure 2 be preserved. This gives rise to two equations of the form:

$$\mu' = h(t, t_W)\mu \quad (7a)$$

$$\mu'_W = l(t, t_W)\mu_W \quad (7b)$$

respectively associated with the transformations of figure 2(c) and 2(f).

Using equations (3), (6) and (7), we obtain the magnetizations M and M_W as functions of the temperature, for typical values of K_W/K. The curves obtained for $q = 2$ are shown in figure 4. We also show in figure 5 the curves for other values of q

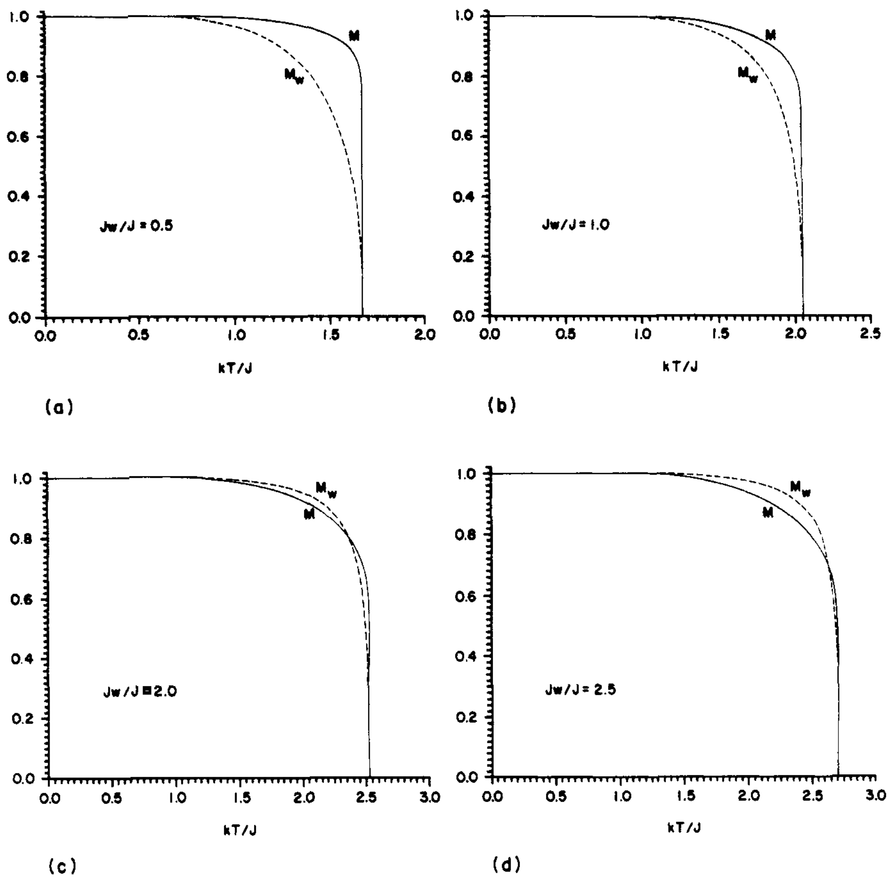

Figure 4. Magnetizations M and M_W as functions of the temperature for $q=2$, (a) $K_W/K=0.5$, (b) $K_W/K=1.0$, (c) $K_W/K=2.0$ and (d) $K_W/K=2.5$. Full curves denote M and broken lines denote M_W.

(3, 4 and 5), for $K_W/K=2$. The associated critical exponents β and β_W are displayed in table 1. They have been analytically calculated using a similar expression as the one which appears in Caride and Tsallis (1987). As we now have two different recurrence equations for K and K_W, it is necessary to calculate the eigenvalues of the Jacobian of the transformation at the fixed critical points, and then generalize the expression for β and β_W:

$$\beta \sim \frac{\ln[1/h(t^c, t_W^c)]}{\ln \lambda_>/(t^c, t_W^c)} \tag{8a}$$

$$\beta_W \sim \frac{\ln[1/l(t^c, t_W^c)]}{\ln \lambda_>/(t^c, t_W^c)} \tag{8b}$$

where $\lambda_>$ is the greatest eigenvalue (greater than 1) at the critical fixed point. We also verify, by calculating the exponents through the curves obtained, that both β and β_W, for a fixed q, do not vary with the ratio K_W/K, as it must be on the basis of the concept of universality classes.

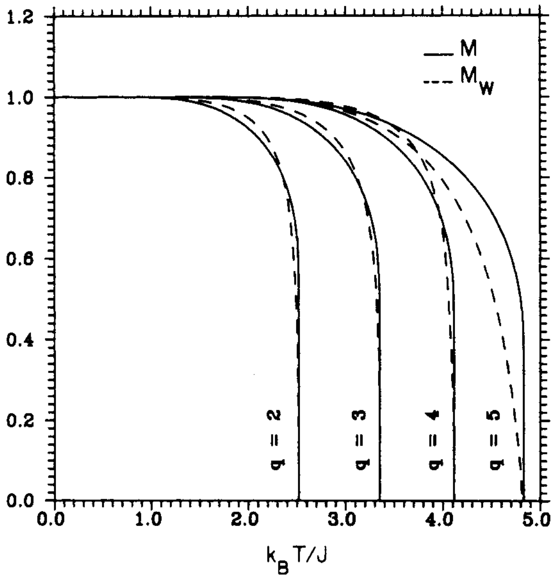

Figure 5. Magnetizations M and M_W as functions of the temperature for $K_W/K = 2.0$ and $q = 2$, 3, 4 and 5. Full curves denote M whereas broken curves denote M_W.

Table 1. Present RG values for the critical exponents β and β_W for $q = 2$, 3, 4 and 5.

q	t	t_W	β	β_W
2	0.7746	0.0328	0.034	0.59
3	0.7791	0.0118	0.0324	0.86
4	0.7814	0.0055	0.032	1.1
5	0.7829	0.0030	0.0319	1.4

When K_W/K is below a certain value, M_W is below M for all temperatures. When K_W/K increases (see the case $K_W/K = 2$ in figure 4), for low temperatures M_W is *above* M and for a given temperature these curves intersect. In this way, M_W always goes to zero at the critical temperature *below* M (since for all K_W/K, $\beta_W > \beta$). We see that the q-evolution of the critical exponents β and β_W suggests that this is true for any value of q, as far as the transition is second order. β does not vary significantly with q (as q increases it slightly decreases) whereas β_W greatly increases as q increases. For $q = 4$ and $q = 5$ we obtain values for β_W greater than unity (see table 1). It is worth stressing that, to the best of our knowledge, this is the first time that a β greater than unity is obtained in a magnetic system.

We could think of M_W as a mean magnetization for the region of internal walls on the carpet. We believe that this region would play an analogous role as a free frontier in a semi-infinite bidimensional system. In this way, it is interesting to compare the M_W behaviour with, for instance, the behaviour of the frontier magnetization of an Ising ferromagnet in a semi-infinite square lattice. The Ising model in the semi-infinite square lattice has been exactly solved by Au-Yang (1973). The interactions on the frontier, say J_W, are allowed to be different from the other interactions, J. As expected, M_W increases as J_W increases. M_W approaches M, the bulk order parameter (see, for

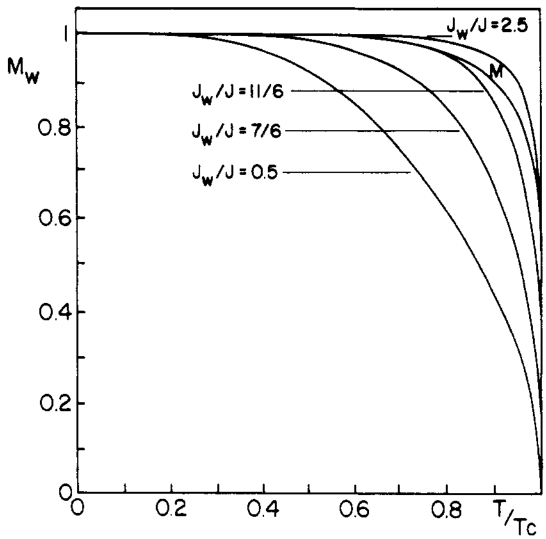

Figure 6. Frontier magnetization M_W as a function of the temperature for the Ising ferromagnet in a semi-infinite square lattice, for some values of J_W/J. The bulk magnetization M is also shown as a reference.

example, McCoy and Wu 1973), when $J_W/J \simeq 2$ (see figure 6). We have used the expressions which appear in the work of Au-Yang to obtain the frontier order parameter M_W for higher values of J_W/J and compared it to M. We find that M_W is *greater* than M for $J_W/J \simeq 2.5$, for instance, and that these curves intersect in very much the same way as the ones we found for the Sierpinski carpet. The values found for the critical exponents β_W and β are respectively 1/2 and 1/8 ($\beta_W > \beta$, as for the carpet).

In summary, we have obtained the approximate behaviour of the magnetization for a Potts ferromagnet on a Sierpinski carpet ($b=3, l=1$). Two typical regions on this carpet have been simulated through suitably chosen hierarchical lattices. We have observed that, for several values of q, the magnetization of the region which consists of internal walls on the fractal has an exponent β_W greater than the exponent β associated with the magnetization of the set of internal sites. The present formalism, as other similar methods, does not allow us to predict first-order transitions which must occur for values of q above a critical value, unknown for a fractal lattice.

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